Увод в теорията на операторни алгебри с приложение в математиката и физиката
Introduction to Theory of Operator Algebras with Applications to Mathematics and Physics
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Lecture 1

I. Преговор по асоциативни алгебри

Def. Complex associative algebra (комплексна асоциативна алгебра)
This is a vector space $\mathcal{A}$ over $\mathbb{C}$ equipped with a bilinear map, the multiplication:

$$\mathcal{A} \times \mathcal{A} \to \mathcal{A}, \qquad (a, b) \mapsto a \cdot b$$

which is associative: $a \cdot (b \cdot c) = (a \cdot b) \cdot c \quad \forall a, b, c \in \mathcal{A}$.

Bilinearity means:

$$(\alpha a + \beta b) \cdot c = \alpha a \cdot c + \beta b \cdot c \qquad \forall \alpha, \beta \in \mathbb{C},$$
$$c \cdot (\alpha a + \beta b) = \alpha c \cdot a + \beta c \cdot b \qquad \forall a, b, c \in \mathcal{A},$$

which include in particular the distributive lows.

Main example: Let V be a vector space over $\mathbb{C}$,

$$\mathrm{End}_{\mathbb{C}} V \left(=: \mathrm{End}\, V \right) := \{ a : V \to V \mid a \text{ is linear} \}$$

$$\Big(\underset{\substack{\cap\\ \mathbb{C}}}{\alpha}\, \underset{\substack{\cap\\ \mathcal{A}}}{a} + \underset{\substack{\cap\\ \mathbb{C}}}{\beta}\, \underset{\substack{\cap\\ \mathcal{A}}}{b}\Big) [\underset{\substack{\cap\\ V}}{v}] := \alpha\, a(v) + \beta\, b(v), \qquad \Big(\underset{\substack{\cap\\ \mathcal{A}}}{a} \cdot \underset{\substack{\cap\\ \mathcal{A}}}{b}\Big) [\underset{\substack{\cap\\ V}}{v}] := a(b(v))$$

If $V = \mathbb{C}^n$, then $\mathrm{End}\, V = \mathrm{Mat}(n, \mathbb{C})$.
"+" is the sum of matrices, "·" is the matrix multiplication.

If $a \cdot b = b \cdot a$ for all $a, b \in \mathcal{A}$ then $\mathcal{A}$ is called commutative (or Abelian) algeba (комутативна или още Абелева).

Def. Morphism (морфизъм) of associative algebras $\varphi : \mathcal{A} \to \mathcal{B}$ is a linear map s.t. (= such that)

$$\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b) \qquad (\forall a, b \in \mathcal{A})$$

If instead: $\varphi(a \cdot b) = \varphi(b) \cdot \varphi(a)$, φ is called antimorphism.

If $\varphi : \mathcal{A} \to \mathcal{B}$ is an antimorphism then it induces a morphism $\mathcal{A} \to \mathcal{B}^{op}$, where for an associative algebra $\mathcal{A}$ the opposite algebra $\mathcal{A}^{op}$ is defined as the same vector space, $\mathcal{A} = \mathcal{A}^{op}$ (as vect. sp.) but with a multiplication $\underset{\mathcal{A}^{op}}{\bullet}$

$$a \underset{\mathcal{A}^{op}}{\bullet} b := b \underset{\mathcal{A}}{\bullet} a$$

The following two notions are equivalent:

<u>Def.</u> Representation (представяне) ≡ Left module (ляв модул)

is a morphism: $\pi : \mathcal{A} \to \operatorname{End} V$ ≡ is a bilinear map:

$$\mathcal{A} \times V \to V$$

$$(a, v) \mapsto a \cdot v ,$$

which is associative:

$$\pi(a) \cdot \pi(b) = \pi(a \cdot b) \quad \underset{\uparrow}{\Longleftrightarrow} \quad a \cdot (b \cdot v) = (a \cdot b) \cdot v$$

under the identification $\pi(a)[v] = a \cdot v$

There are analogous "right" notions:

"Anti-representation" ≡ Righ module (десен модул)

is an antimorphism: $\pi : \mathcal{A} \to \operatorname{End} V$ ≡ is a bilinear map:

$$V \times \mathcal{A} \to V$$

$$(v, a) \mapsto v \cdot a ,$$

which is associative:

$$\pi(a) \cdot \pi(b) = \pi(b \cdot a) \quad \underset{\uparrow}{\Longleftrightarrow} \quad (v \cdot a) \cdot b = v \cdot (a \cdot b)$$

under the identification $\pi(a)[v] = a \cdot v$

<u>Def.</u> Left / right ideal (ляв/десен идеал) in an assoc. algebra $\mathcal{A}$ is a vector subspace $\mathcal{J} \subseteq \mathcal{A}$ s.t.

$\forall a \in \mathcal{A}$: $\quad a \cdot b \in \mathcal{J}$, $\quad b \cdot a \in \mathcal{J}$

$\forall b \in \mathcal{J} \quad$ (left) $\quad$ (right)

The latter are shortly abbreviated by:

$$\mathcal{A} \cdot \mathcal{J} \subseteq \mathcal{J} , \quad \mathcal{J} \cdot \mathcal{A} \subseteq \mathcal{J} \quad \text{(respectively)}.$$

If $J \subseteq \mathcal{A}$ is a left ideal then the quotient space $\mathcal{A}/J$ (фактор пространство) has a natural structure of a left module: denoting the elements: $[c]_J \equiv c + J \in \mathcal{A}/J$ for $c \in \mathcal{A}$ we set:

$$a \cdot [c]_J := [a \cdot c]_J \quad \text{for all } a \in \mathcal{A}.$$

Correctness:

indeed:

$$\begin{tikzcd}
{[c]_J = [c']_J} \arrow[r, Rightarrow, "?"] \arrow[d, Leftrightarrow, "\text{by def.}"] & {a \cdot [c]_J = a \cdot [c']_J} \arrow[d, Leftrightarrow, "\text{by def.}"] \\
{c - c' \in J} \arrow[r, Rightarrow] & {a \cdot c - a \cdot c' = a(c - c') \in J}
\end{tikzcd}$$

<u>Example</u> Let V be a representation of $\mathcal{A}$, $v \in V$. Set:
$\mathrm{Ann}(v) := \{ a \in \mathcal{A} \mid \pi(a)[v] = 0 \}$ - this is a left ideal (check!) called annihilation ideal (анулатор) of v.

<u>Question.</u> For a given left ideal $J \subseteq \mathcal{A}$ is there a representation V of $\mathcal{A}$ and $v \in V$ s.t. $J = \mathrm{Ann}(v)$?

If the algebra $\mathcal{A}$ has a unit - yes. Before explainig this:

<u>Def.</u> A unit in a assoc. algebra $\mathcal{A}$: $1_{\mathcal{A}} \, (=: 1) \in \mathcal{A}$ s.t. $1 \cdot a = a \cdot 1 = a \quad (\forall a \in \mathcal{A})$.
Algebra with unit (алгебра с единица) is also called unital algebra.
Morphism of unital algebras (морфизъм на алгебри с единица) is a morphism $\varphi : \mathcal{A} \to \mathcal{B}$ s.t. in addition $\varphi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$.
A representation of unital algebra is a morphism of unital algebras i.e. $\pi(1_{\mathcal{A}}) = 1_{\mathrm{End}\,V} \equiv \mathrm{id}_V$ - the identity map $\mathrm{id}_V : V \to V$.

<u>Precaution</u>: Given a morphism $\varphi : \mathcal{A} \to \mathcal{B}$ only of assoc. algebras which have units $1_{\mathcal{A}}$, $1_{\mathcal{B}}$ we have:

$$\varphi(1_{\mathcal{A}})\,\varphi(a) = \varphi(1_{\mathcal{A}} \cdot a) = \varphi(a) = \varphi(a \cdot 1_{\mathcal{A}}) = \varphi(a) \cdot \varphi(1_{\mathcal{A}})$$

Can we conclude then that $\varphi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$?

Counterexample: Let $m < n$ and consider $\mathrm{Mat}(m, \mathbb{C}) \to \mathrm{Mat}(n, \mathbb{C}) : A \mapsto \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$

Let us return to the answer of the above question. Take $V := \mathcal{A}/J$ and $v = [1]_J$. Then:

$$\operatorname{Ann}[1]_J = \{ a \in \mathcal{A} \mid a \cdot [1]_J = 0 \} = J \quad \text{since:}$$

$$a \cdot [1]_J = [a]_J = 0 \overset{}{\underset{\text{by def.}}{\Longleftrightarrow}} a \in J .$$

The element $[1]_J$ has one additional important property:

<u>Def.</u> Algebraically cyclic vector (алгебричен цикличен вектор) $v \in V$ in a representation $\pi : \mathcal{A} \to \operatorname{End} V$ is such a vector that $\pi(\mathcal{A}) . v := \{ \pi(a)(v) \mid a \in \mathcal{A} \} = V$. A representation having a cyclic vector is called cyclic representation (циклично представяне).

<u>Proposition 1.1.</u> Let V be a representation of a unital algebra $\mathcal{A}$ which has a cyclic vector $v \in V$. Then if $J = \operatorname{Ann}(v)$ there is a unique linear map $F : \mathcal{A}/J \to V$ such that

$$F\big([a]_J\big) = \pi(a)(v) \quad (\forall a \in \mathcal{A}) .$$

Furthermore, F is an isomorphism and the following diagram is commutative

$$\begin{tikzcd}
\mathcal{A}/J \arrow[r, "\rho(a)"] \arrow[d, "F"', "\wr"] & \mathcal{A}/J \arrow[d, "F", "\wr"'] \\
V \arrow[r, "\pi(a)"'] & V
\end{tikzcd}
\qquad \text{i.e.} \quad F \circ \rho(a) = \pi(a) \circ F \quad \text{for all } a \in \mathcal{A}$$

where $\rho(a)$ stands for the canonical representation of $\mathcal{A}$ on $\mathcal{A}/J$

$$\rho(a)\big([c]_J\big) := [a \cdot c]_J .$$

<u>Proof.</u> Clearly, F is unquely defined by the identity $F\big([a]_J\big) = \pi(a)(v)$ $(\forall a \in \mathcal{A})$ provided that it is correct, i.e. that:

indeed:

$$\begin{tikzcd}
{[a]_J = [a']_J} \arrow[r, Rightarrow, "?"] \arrow[d, Leftrightarrow, "\text{by def.}"] & \pi(a)(v) = \pi(a')(v) \arrow[d, Leftrightarrow, "\text{by def.}"] \\
a - a' \in J = \operatorname{Ann}(v) \arrow[r, Leftrightarrow] & \pi(a)(v) - \pi(a')(v) = \pi(a-a')(v) = 0 .
\end{tikzcd}$$

Furthermore, since above we had equivalence, then

$[a]_{\mathcal{J}} = [a']_{\mathcal{J}} \iff F([a]_{\mathcal{J}}) = F([a']_{\mathcal{J}})$ i.e. F is injective

(F е инекция = инективно изображение).

F is also surjection (сюрекция) since
Image of $F = \pi(\mathcal{A})(v) = V$ because v is cyclic vector.
Thus, F is an isomorphism.
The diagram is left for an exercise (hint: apply both sides of the identity $F \circ \rho(a) = \pi(a) \circ F$ to an arbitrary $[c]_{\mathcal{J}} \in \mathcal{A}/\mathcal{J}$). $\square$

The diagram in Proposition 1.1. is an example of the following general notion.

<u>Def.</u> Let $\pi : \mathcal{A} \to \operatorname{End} V$ and $\rho : \mathcal{A} \to \operatorname{End} W$ be two representations of an associative algebra $\mathcal{A}$. A linear map $F : V \to W$ is called a splitting map (сплитащо изображение)
or also: a morphism of representations, $\mathcal{A}$-linear map
iff (= if and only if) the following diagram is commutative

$$\begin{tikzcd}
V \arrow[r, "\pi(a)"] \arrow[d, "F"'] & V \arrow[d, "F"] \\
W \arrow[r, "\rho(a)"'] & W
\end{tikzcd}$$

i.e.: $F \circ \pi(a) = \rho(a) \circ F \quad \forall a \in \mathcal{A}$.

If F is an isomorphism the representations are called isomorphic.

Now the meaning of Proposition 1.1. is that any cyclic representation of a unital algebra $\mathcal{A}$ <u>up to an isomorphism</u> (с точност до изоморофизъм) is of the form $\mathcal{A}/\mathcal{J}$ for some left ideal $\mathcal{J} \subseteq \mathcal{A}$.

Corollary 1.2. Let V and W be two representations of a unital algebra $\mathcal{A}$, which have cyclic vectors $v \in V$ and $w \in W$.
Then there exists a unique isomorphism $F : V \cong W$ of the two representations s.t. $F(v) = w$

iff $\quad \mathrm{Ann}(v) = \mathrm{Ann}(w)$.

Def. Two-sided ideal = ideal (двустранен идеал = идеал) $\mathcal{J} \subseteq \mathcal{A}$ in an associative algebra $\mathcal{A}$ is a left and right ideal.

Then $\mathcal{A}/\mathcal{J}$ has a structure of a complex associative algebra and the canonical projection $p : \mathcal{A} \to \mathcal{A}/\mathcal{J}$ is a morphism of assoc. algebras,

$$[a]_{\mathcal{J}} \cdot [b]_{\mathcal{J}} = [a \cdot b]_{\mathcal{J}}$$

Proposition 1.3. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a morphism of complex associative algebras. Then $\mathrm{Ker}\,\varphi := \{ a \in \mathcal{A} \mid \varphi(a) = 0 \}$ is an ideal.

If σ is surjective map then $\exists!$ isomorphism $\theta : \mathcal{A}/\mathrm{Ker}\,\sigma \cong \mathcal{B}$

s.t.
$$\begin{tikzcd} \mathcal{A} \arrow[r, "p"] \arrow[dr, "\varphi"'] & \mathcal{A}/\mathrm{Ker}\,\sigma \arrow[d, "\wr"', "\theta"] \\ & \mathcal{B} \end{tikzcd}$$
is commutative, i.e. $\varphi = \theta \circ p$, where $p(a) = [a]_{\mathcal{J}}$ is the canonical projection.

Sketch of the Proof Check as an exercise that $\mathrm{Ker}\,\varphi$ is an ideal.
The identity $\varphi(a) = \theta \circ p(a)$ means $\varphi(a) = \theta([a]_{\mathcal{J}})$ since $p(a) = [a]_{\mathcal{J}}$ by definition. Then $\theta([a]_{\mathcal{J}}) = \varphi(a)$ uniquely determines θ provided that this is correct, i.e. if

$[a]_{\mathcal{J}} = [a']_{\mathcal{J}} \overset{?}{\Longrightarrow} \varphi(a) = \varphi(a')$ this is verified just as in Proposition 1.1. (check!) and we have even more:
$[a]_{\mathcal{J}} = [a']_{\mathcal{J}} \Longleftrightarrow \varphi(a) = \varphi(a')$ i.e. θ is an injection.
θ is surjection since φ is surjection.
Finally, check that θ is a morphism: $\theta(a \cdot b) \overset{?}{=} \theta(a) \cdot \theta(b)$. $\square$

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Remark: relation to the theory of representation of groups.

Group algebra (групова алгебра) for a finite group G.

Consider $\mathbb{C}^G := \left\{ \sum_{g\in G} x_g \vec{e}_g \mid x_g \in \mathbb{C} \ \forall g \in G \right\} \cong \mathbb{C}^{|G|}$

$$\vec{e}_g \cdot \vec{e}_h := \vec{e}_{g\cdot h} \Rightarrow \left(\sum_{g\in G} x_g \vec{e}_g\right) \cdot \left(\sum_{h\in G} y_h \vec{e}_h\right) =$$

$$= \sum_{g\in G} \sum_{h\in G} x_g y_h \vec{e}_g \cdot \vec{e}_h = \sum_{g\in G} \left(\sum_{\substack{g_1, g_2 \in G \\ \text{s.t. } g = g_1 g_2}} x_{g_1} y_{g_2} \right) \vec{e}_g$$

associativity follows from the group associativity – check!

Then we have the following equivence:

Group representation		Rerpesentation of the group algebra $\mathbb{C}^G$.
$\pi_0 : G \to \operatorname{End} V$	$\Longleftrightarrow$	
$\pi_0(g) = \pi(\vec{e}_g)$		$\pi\left(\sum_{g\in G} x_g \vec{e}_g\right) = \sum_{g\in G} x_g \pi_0(g)$

II Преговор на началните понятия от теория на Банахови пространства

Normed space (нормирано пространство) is a complex vector space V equipped with a real valued function $\|\cdot\| : V \to [0,\infty)$ called norm, s.t.:

(i) $\|\alpha v\| = |\alpha| . \|v\|$, $\forall \alpha \in \mathbb{C}$, $\forall v \in V$

(ii) $\|v + w\| \le \|v\| + \|w\|$, $\forall v, w \in V$

(iii) $\|v\| = 0 \Leftrightarrow v = 0$.

Then the function $d : V \times V \longrightarrow [0,\infty)$, $d(v,w) := \|v - w\|$ is a metric (метрика) on V:

$$d(v,t) \le d(v,w) + d(w,t) \;;\; d(v,w) = 0 \Leftrightarrow v = w .$$

Banach space (Банахово пространство) is a normed space that is complete metric space w.r.t. (with respect to) d.

The completeness means:
if $\{v_k\}_{k=1}^{\infty}$ is a Cauchy's fundamental sequence (фундаментална редица / редица на Коши)
i.e. if $\|v_j - v_k\| \xrightarrow[j,k\to\infty]{} 0$, then $\exists v \in V$ s.t. $\|v_k - v\| \xrightarrow[k\to 0]{} 0$.

It follows that this v is unique and we write $v = \lim\limits_{k\to\infty} v_k$.

Another useful inequality in a normed space is:

$$\big|\,\|v\| - \|w\|\,\big| \le \|v - w\| \quad (\forall v, w \in V) \qquad \left(\left.\begin{array}{l} \|v\| - \|w\| \\ \|w\| - \|v\| \end{array}\right\} \le \|v - w\| = \|w - v\| \right)$$

This implies that the norm is a continuous (непрекъсната) function:
if $v_k \xrightarrow[k\to\infty]{} v$ then $\lim\limits_{k\to\infty} \|v_k\| = \|v\|$ since

$$\big|\,\|v_k\| - \|v\|\,\big| \le \|v_k - v\| \xrightarrow[k\to\infty]{} 0 .$$

Similarly, if $\{v_k\}_{k=1}^{\infty}$ is fundamental then $\exists \lim\limits_{k\to\infty} \|v_k\|$.

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Example: $\mathbb{C}$ is a Banach space

Main construction of Banach spaces $\mathcal{F} = \text{Bounded Maps}(M, V)$:

Proposition 1.4. Let M be a set, V - Banach space and $\mathcal{F} := \{ F : M \to V \mid F \text{ is bounded map, i.e. } \|F\| < \infty \}$,
(ограничено изображение)

where $\|F\| := \sup\limits_{p \in M} \|F(p)\|$ and $\mathcal{F}$ is endowed with the vector space structure:

$$\Big(\underset{\mathbb{C}}{\underset{\cap}{\alpha_1}} \underset{\mathcal{F}}{\underset{\cap}{F_1}} + \underset{\mathbb{C}}{\underset{\cap}{\alpha_2}} \underset{\mathcal{F}}{\underset{\cap}{F_2}}\Big)(\underset{M}{\underset{\cap}{p}}) := \alpha_1 F_1(p) + \alpha_2 F_2(p).$$

Then $\mathcal{F}$ is a Banach space.

Proof. $\|F_1 + F_2\| = \sup\limits_p \|F_1(p) + F_2(p)\| \le \sup\limits_p \big(\|F_1(p)\| + \|F_2(p)\|\big) \le$
$\le \sup\limits_p \|F_1(p)\| + \sup\limits_p \|F_2(p)\| \le \|F_1\| + \|F_2\|$. The remainig properties of the norm are similarly verified.

Completeness: Let $F_k : M \to V$ be a sequence of bounded maps, i.e. $F_k \in \mathcal{F}$ and $\|F_j - F_k\| \to 0$ - Cauchy's sequence.

Then:

$$\begin{array}{ccc} \|F_k(p) - F_j(p)\| & \le & \|F_k - F_j\| \\ \downarrow{\scriptstyle j,k\to\infty} & \Leftarrow & \downarrow{\scriptstyle j,k\to\infty} \\ 0 & & 0 \end{array} \quad (\forall p \in M) \Rightarrow \{F_k(p)\}_{k=1}^{\infty} \text{ is a Cauchy's seq. in } V$$

$$\Rightarrow \exists \lim_{k\to\infty} F_k(p) =: F(p).$$

Why F is bounded, i.e. $\|F\| \overset{?}{<} \infty$, and why $\|F_k - F\| \xrightarrow[k\to\infty]{} 0$?

$\|\cdot\|$ is continuous, $\{F_k\}_{k=1}^{\infty}$ is fundamental $\Rightarrow \exists \lim\limits_{k\to\infty} \|F_k\|$.

Then: $\|F\| = \sup\limits_p \|F(p)\| = \sup\limits_p \lim\limits_{k\to\infty} \|F_k(p)\| \overset{!}{\le} \lim\limits_{k\to\infty} \sup\limits_p \|F_k(p)\| = \lim\limits_{k\to\infty} \|F_k\| < \infty$

Next: $\|F_k - F\| = \sup\limits_p \|F_k(p) - F(p)\| = \sup\limits_p \|F_k(p) - \lim\limits_{j\to\infty} F_j(p)\| =$

$= \sup\limits_p \lim\limits_{j\to\infty} \|F_k(p) - F_j(p)\| \le \lim\limits_{j\to\infty} \sup\limits_p \|F_k(p) - F_j(p)\| =$

$= \lim\limits_{j\to\infty} \|F_k - F_j\| \xrightarrow[k\to\infty]{} 0.$

□

Def. Normed associative algebra (асоциативна алгебра с норма) is a complex associative algebra $\mathcal{A}$, which is also a normed space s.t.

$$\|a\cdot b\| \le \|a\|\cdot\|b\| \quad (\forall a,b\in\mathcal{A}).$$

Banach algebra (Банахова алгебра) is a normed algebra that is a Banach space.

Corollary 1.5. In a normed algebra $\mathcal{A}$ if $a_k \xrightarrow[k\to\infty]{} a$, $b_k \xrightarrow[k\to\infty]{} b$ then $a_k\cdot b_j \xrightarrow[j,k\to\infty]{} a\cdot b$.

Proof. $\|a_j\cdot b_k - a\cdot b\| = \|a_j\cdot b_k - a\cdot b_k + a\cdot b_k - a\cdot b\| \le$

$$\le \|a_j - a\| \underbrace{\|b_k\|}_{\substack{\text{bounded sequence}\\ \text{(why ?)}}} + \|a\|\,\|b_k - b\| \xrightarrow[j,k\to\infty]{} 0. \qquad \square$$

Def. A linear operator $A: V\to V$ on a Banach space is called bounded (ограничен) iff

$\exists\, \|A\| := \sup\{\|A(v)\| \mid v\in V,\ \|v\| = 1\} < \infty$ (notation: $\|A\|<\infty$).

Corollary 1.6. $\|A(v)\| \le \|A\|\,\|v\|$ $\forall A$ with $\|A\|<\infty$ and $\forall v\in V$.

Proposition 1.7. Let $\mathcal{B}(V) := \{A : V \xrightarrow[\text{lin}]{} V \mid \|A\| < \infty\} \subseteq \operatorname{End} V$ for a Banach space V. Then $\mathcal{B}(V)$ is a Banach algebra.

Proof. Let $B_1 := \{v\in V \mid \|v\| = 1\}$ - the unit sphere

Every $A\in\mathcal{B}(V)$ is uniquely determined by its restriction $A|_{B_1}$ since $A(v) = \|v\|\, A\left(\frac{v}{\|v\|}\right)$ for $v\ne 0$. Then it follows that

$\mathcal{B}(V) \to \text{Bounded Maps}(B_1, V)$ is a linear injection,

$A \mapsto A|_{B_1}$

which preserves the norms (by the norm defintions).

Thus, $\mathcal{B}(V)$ is a normed space.

It remains to prove that the image of $\mathcal{B}(V)$ in $\text{Bounded Maps}(B_1, V)$ is a closed subspace.

Let $A_k \in \mathcal{B}(V)$ be a sequence of bounded operators such that $\|A_j - A_k\| \xrightarrow[j,k \to \infty]{} 0$. Then for all $v \in V$:

$\|A_j(v) - A_k(v)\| \leq \|A_j - A_k\| \|v\| \xrightarrow[j,k \to \infty]{} 0$, so $A_k(v)$ is convergent on the whole V, $A(v) = \lim\limits_{k \to \infty} A_k(v)$ is then a linear map. But on B_1: $A_k|_{B_1} \to A|_{B_1}$, so the limit of $A_k|_{B_1}$ is again a restriction of a linear map.

Hence, $A \in \mathcal{B}(V)$ and $\mathcal{B}(V)$ is complete.

$\mathcal{B}(V)$ is a normed algebra:

$$\|A \cdot B\| = \sup_{\|v\|=1} \|A(B(v))\| \leq \|A\| \sup_{\|v\|=1} \|B(v)\| = \|A\| \|B\|. \quad \square$$

<u>Notation</u> Usually the convergence in $\mathcal{B}(V)$ is denoted by:

$A_k \rightrightarrows A \quad \left(\overset{}{\underset{def}{\Longleftrightarrow}} \|A_k - A\| \xrightarrow[k \to \infty]{} 0 \right).$

Similarly, the convergence in $\text{Bounded Maps}(M, V)$ is denoted by $F_k \rightrightarrows F \quad \left(\underset{def}{\Longleftrightarrow} \|F_k - F\| \xrightarrow[k \to \infty]{} 0 \right)$.

Both are called uniform convergence (равномерна сходимост).

A direct poof of Proposition 1.7.

$\mathcal{B}(V)$ is normed space: $\|A+B\| = \sup\limits_{\|v\|=1} \|A(v)+B(v)\|$

$\leq \sup\limits_{\|v\|=1} \left(\|A(v)\| + \|B(v)\|\right) \leq \sup\limits_{\|v\|=1} \|A(v)\| + \sup\limits_{\|v\|=1} \|B(v)\| = \|A\| + \|B\|$

. . .

Completeness: let $\|A_j - A_k\| \to 0$ Cauchy sequence (редица на Коши).

Then $\forall v \in V \quad \|A_j(v) - A_k(v)\| \leq \|A_j - A_k\| \, \|v\| \to 0$ - Cauchy sequence

$\Rightarrow \exists!\ A(v) := \lim\limits_{k\to\infty} A_k(v)$ - linear operator on V

From $\big|\|A_j\| - \|A_k\|\big| \leq \|A_j - A_k\| \to 0 \quad \Rightarrow \exists \lim\limits_{k\to\infty} \|A_k\|$

$\Rightarrow \quad \|A(v)\| = \|\lim\limits_{k\to\infty} A_k(v)\| = \lim\limits_{k\to\infty} \|A_k(v)\| \leq \lim\limits_{k\to\infty} \|A_k\| \underbrace{\|v\|}_{=1} < \infty$

for all $\|v\| = 1$, i.e. $A \in \mathcal{B}(V)$

Why $\|A - A_k\| \to 0$: $\|A - A_k\| = \sup\limits_{\|v\|=1} \|(A - A_k)(v)\| = \sup\limits_{\|v\|=1} \lim\limits_{j\to\infty} \|(A_j - A_k)(v)\|$

$\leq \lim\limits_{j\to\infty} \sup\limits_{\|v\|=1} \|(A_j - A_k)(v)\| = \lim\limits_{j\to\infty} \|A_j - A_k\| \xrightarrow[k\to\infty]{} 0. \quad \square$